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TANGENTIALLY DISTAL FLOWS

BY

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ABSTRACT

Using a modified version of Furstenberg's suggested definition of tangential distality, a differentiable analogue of the Furstenberg Structure Theorem for distal transformation groups is proved. Examples are given which show tangential distality is of a rather different character to distality. Conditions are given under which tangential distality implies distality.

Let X be a compact C^1 manifold, and $f: X \to X$ a diffeomorphism. Any two Riemannian structures on X are *equivalent*, in the sense that, if $\| \ \|_1$, $\| \ \|_2 : T_*X \to \mathbf{R}^+$ are the associated "norms" on T_*X, then there exist $m, M > 0$ such that:

$$m \| v \|_1 \leqq \| v \|_2 \leqq M \| v \|_1 \qquad \text{for all } v \in T_*X.$$

Furstenberg suggested the following definition: Suppose a Riemannian structure on X has been given, with associated norm $\| \ \|$. $f: X \to X$ is said to be *tangentially distal* if:

$$0 < \inf_n \| Df^n v \| \qquad \text{for all } v \in T_*X, \quad v \neq 0.$$

(By the above, the choice of norm does not affect the definition.)

One might expect the property of tangential distality to be related to that of distality: Let d be any metric for the topology of X, and recall that $f: X \to X$ is *distal* if:

$$\inf_n d(f^n x, f^n y) > 0 \qquad \text{for all } x, y \in X, \quad x \neq y.$$

In fact, there is a class of examples which are both distal and tangentially distal: a typical example is a skew-product diffeomorphism of an n-torus (1.4).

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However, even for the circle, distal diffeomorphisms need not be tangentially distal (1.5). Tangentially distal *flows* which are not distal include the classical horocycle flow (1.11), and change of velocity flows of certain distal flows (1.8). Another class of tangentially distal flows, which might (or might not) exist, is discussed briefly in 1.12. The existence or non-existence of such examples might indicate how great are the restrictions on a manifold which is the phase space of a minimal tangentially distal transformation group — the restrictions in the case of minimal distal transformation groups are quite severe.

One would expect a structure theorem for tangentially distal transformation groups akin to the Furstenberg structure theorem for distal transformation groups. We give such a structure theorem (§2) — the structure is on the tangent bundle, not the manifold itself — under additional (necessary) assumptions.

In §3 we prove that tangentially distal diffeomorphisms satisfying mild additional assumptions have all characteristic-exponents zero, hence have zero entropy. It is known that distal homeomorphisms have zero entropy [18].

In general, the structure for a tangentially distal diffeomorphism (consisting of a finite sequence of C^0 subvector bundles in the tangent bundle) does not arise from a sequence of foliations of the manifold. But if it does, we give, in §3, additional conditions under which a tangentially distal diffeomorphism must be distal.

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§1. Definitions of tangential distality, and some examples

First, we give some motivation for the definitions. Let $T: V \to V$ be an invertible linear transformation of a real finite-dimensional vector space. Let $\bigotimes^k V$ and $\wedge^k V$ denote the kth tensor and exterior products of V respectively.

T is *distal* if, whenever $v \in V$ is such that 0 is a limit point of $\{T^n v : n \in \mathbf{Z}\}$, then $v = 0$. Clearly, T is distal if and only if all the (complex) eigenvalues of T have modulus 1.

It follows that, if T is distal:

(a) $\bigotimes^r T: \bigotimes^r V \to \bigotimes^r V$ is distal for all r.

(b) If W is a subspace of V with $TW = W$, then the induced map $T_W : V/W \to V/W$ is distal.

Combining (a) and (b), we see that:

$$\wedge^k T: \wedge^k V \to \wedge^k V \text{ is distal,} \quad \text{for } 1 \leqq k \leqq n = \dim V.$$

We shall see later that the corresponding result for the derivative of a diffeomorphism is not true. Note that Jordan decomposition of T over the real field gives a type of "Furstenberg Structure Theorem" for distal T.

All the theory of §§1 and 2 can be extended to arbitrary transformation groups with metric phase spaces without any trouble, but for ease of notation we consider only the case of a homeomorphism f of a compact metric space X. In the examples, however, we also consider flows.

1.1. Definition. Let V be a real finite-dimensional C^0 vector bundle over a compact metric space X, and let $f_v : V \to V$ have the following properties relative to a homeomorphism $f : X \to X$:

(i) The following diagram commutes, where $\pi : V \to X$ is the natural projection:

$$\begin{array}{ccc} V & \xrightarrow{f_v} & V \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow[f]{} & X \end{array}$$

(ii) $f_v \,|\, \pi^{-1}x : \pi^{-1}x \to \pi^{-1}fx$ is a linear isomorphism.

We call (V, f_v) an *extension* of (X, f) (e.g., X is a C^1 manifold, f a C^1 diffeomorphism, $V = T_*X$, the tangent bundle of X, and $f_v = Df$, the derivative of f).

Let $\|\ \ \| : V \to \mathbf{R}^+$ be a continuous function which is a norm restricted to each fibre over X (any two such are *equivalent* as in the *introduction*). Then (V, f_v) is a *distal extension* of (X, f) if $\inf_n \|f_v^n w\| > 0$ for all $w \in V$, $w \neq 0$.

1.2. Definition. For $1 \leqq k \leqq n = \dim V$ (where $\dim V$ denotes the dimension of all the fibres $\{V_x : x \in X\}$ of V over X), let $\wedge^k V$ denote the C^0 vector bundle over X, obtained from V, which has $\wedge^k(V_x)$, the kth exterior product of V_x, as fibre over $x \in X$.

Then $\wedge^k(f_v) : \wedge^k V \to \wedge^k V$ is an extension of $f : X \to X$.

Write $\wedge^k X$ for $\wedge^k(T_*X)$, and $\wedge^k f$ for $\wedge^k(Df)$, so that $\wedge^1 X = T_*X$, and $\wedge^1 f = Df$, if X is a C^1 manifold, and f a C^1 diffeomorphism.

1.3. Definition. $f : X \to X$ is *tangentially distal of order* k if $(\wedge^k X, \wedge^k f)$ is a distal extension of (X, f), where X is a compact C^1 manifold and f a C^1 diffeomorphism. So tangential distality of order 1 coincides with Furstenberg's definition of tangential distality.

It is clear that tangential distality of order k is preserved under C^1 conjugation.

EXAMPLES

1.4. Let T^n denote the n-dimensional torus, and let $f: T^n \to T^n$ be a skew-product diffeomorphism, i.e. an additive notation:

$$f(x_1, \cdots, x_n) = (x_1 + \alpha, x_2 + g_2(x_1), \cdots, x_n + g_n(x_1, \cdots, x_{n-1}))$$

where $\alpha \in T^1$ and $g_i \in C(T^{i-1}, T^1)$.

Then for all $x \in T^n$, Df_n is of the form:

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ X & & 1 \end{pmatrix}$$

It is clear that f is tangentially distal of all orders. Note also that f is *distal.*

1.5. Let f be a positively oriented C^2 diffeomorphism of the circle, with irrational rotation number α. Then, by Denjoy's theorem [10], f is C^0 conjugate to the notation:

$$R_\alpha : x \mapsto x + \alpha \qquad (x \in \mathbf{R}/\mathbf{Z} = T^1).$$

However, in general, f need not be C^1-conjugate to R_α [2], [10]. Herman has shown [10] that f is C^1-conjugate to R_α if and only if:

$$0 < \inf_{n,x} Df^n(x) \leqq \sup_{n,x} Df^n(x) < \infty.$$

By 2.2, this is equivalent to tangential distality.

1.6. More generally than 1.4, let (X, T) be a minimal distal transformation group with X a compact C^1 manifold and T a group of C^1 diffeomorphisms. In this case [22] the Furstenberg Structure Theorem implies there exists $\{(X_i, T)\}_{i=0}^r$ $(r \leqq \dim X)$ such that (X_0, T) is trivial, and $(X_i, T) <_{\pi_{i+1}} (X_{i+1}, T)$ $(0 \leqq i \leqq r-1)$, (X_{i+1}, T) being a quotient Lie group extension of (X_i, T).

If we assume in addition that the associated fibre bundle with base X_i and total space X_{i+1} is C^1 (so that, in particular, X_i, X_{i+1} are C^1 and π_{i+1} is a C^1 submersion) then (X, T) is tangentially distal of all orders.

1.7. If $(X, \{T_t\})$ is a C^1 flow on a compact C^1 manifold with no fixed points, so that $v(x)$ is non-vanishing, where

$$v(x) = \frac{d}{dt}\bigg|_{t=0} T_t x,$$

then $DT_t(v(x)) = v(T_t x)$, and V is a C^0 subvector bundle of $T_* X$, invariant under DT_t for all t, where V is spanned by $\{v(x): x \in X\}$.

$(V, \{DT_t\})$ is a distal extension of (X, T_t), since

$$0 < \inf_x \| v(x) \| \leqq \sup_x \| v(x) \| < \infty.$$

1.8. *Changing velocity preserves tangential distality*

Let $(X, \{W_t\})$ be a C^2 flow which is tangentially distal of *all orders*. Let $(X, \{S_t\})$ be obtained from $(X, \{W_t\})$ by C^1 change of velocity, i.e. if w, s are the infinitesimal generators of $\{W_t\}$, $\{S_t\}$ respectively,

$$s(x) = \frac{1}{k(x)} w(x),$$

where $k : X \to \mathbf{R}$ is continuous and strictly positive.

There exists $C^1 \bar{h} : X \times \mathbf{R} \to X$ with:

$$S_t x = W_{\bar{h}(x,t)} x,$$

$$\frac{\partial \bar{h}}{\partial t}(x, 0) = \frac{1}{k(x)}.$$

We claim that $(X, \{S_t\})$ is tangentially distal of all orders. But if, for example, X is the 2-torus T^2 and $\{W_t\}$ is given by $W_t(x, y) = (x + t, y + \lambda t)$, then $(X, \{S_t\})$ need not be distal, cf. [11], [19].

We give the proof that $(X, \{S_t\})$ is tangentially distal of order 1. The proof for higher orders is similar.

$$DW_t(w(x)) = w(W_t x) \quad \text{and} \quad DS_t(s(x)) = s(S_t x).$$

Thus we only need to show that for $v \in T_x X$, v not a multiple of $s(x)$,

$$\inf_t \| (DS_t v) \wedge s(S_t x) \| > 0.$$

But it can be checked that:

$$(DS_t v) \wedge s(S_t x) = (DW_{\bar{h}(x,t)} v) \wedge s(S_t x)$$

$$= \frac{1}{k(W_{\bar{h}(x,t)} x)} \Lambda^2 (DW_{\bar{h}(x,t)})(v \wedge w(x)),$$

and $\inf_t \| \Lambda^2 (DW_{\bar{h}(x,t)})(v \wedge w(x)) \| > 0$, since

$(X, \{W_t\})$ is tangentially distal of order 2.

1.9. Let X be a compact n-dimensional C^1 manifold, and $f:X\to X$ a minimal C^1 diffeomorphism. A necessary and sufficient condition for (X,f) to be tangentially distal of order n is that there exists a continuous f-invariant measure on X.

For sufficiency, let $\omega : U\to \wedge^n U$ be the continuous n-form on an open subset U of X, associated to μ. By the f-invariance of μ, if x, $f^m x\in U$,

$$\wedge^n(f^m)(\omega(x))=\omega(f^m x).$$

Since (X,f) is minimal, for each x, $\{m : f^m x\in U\}$ is syndetic, hence for each x,

$$\inf_m \wedge^n(f^m)(\omega(x))>0,$$

as required.

For necessity, by lifting to a double cover of X, we can assume X is orientable. Let ω_0 be a non-vanishing section: $X\to\wedge^n X$.

We wish to find another continuous non-vanishing section ω_1 such that:

$$\wedge^n(f)(\omega_1(x))=\varepsilon\omega_1(fx)\qquad\text{for all } x\in X,$$

where $\varepsilon=\pm1$.

Let $h:X\to\mathbf{R}$ be such that:

$$\wedge^n(f)(\omega_0(x))=h(x)\omega_0(fx),$$

where either $h>0$ or $h<0$. So

$$|h(x),\cdots,h(f^{m-1}x)|\,\|\omega_0(f^m x)\|=\|\wedge^n(f^m)(\omega_0(x))\|.$$

So $0<\inf_m|h(x),\cdots,h(f^{m-1}x)|\le\sup_m|h(x),\cdots,h(f^{m-1}x)|<\infty$. (For the right-hand inequality, see 2.2.)

But then there exists a continuous function $h_1:X\to\mathbf{R}$ such that

$$\frac{h_1(f(x))}{h_1(x)}=|h(x)|\qquad\text{for all } x.$$

(See [8].) Put $\omega_1=h\cdot\omega_0$.

$$\omega_1(fx)=h_1(fx)\omega_0(fx)=\frac{h_1(fx)}{h(x)}\wedge^n(f)(\omega_0(x))$$

$$=\varepsilon h_1(x)\cdot\wedge^n(f)(\omega_0(x))=\varepsilon\wedge^n(f)(\omega_1(x))$$

for $\varepsilon=\pm1$, as required.

1.10. In what follows, let $g : T^2 \to T^2$ be of the form:

$$g(x, y) = (f(x), x + y)$$

where f is a positively oriented C^2 diffeomorphism of the circle with irrational rotation number α, in 1.5; (T^2, g) is minimal. We see that

$$Dg^n_{x,y} = \begin{pmatrix} Df^n(x) & 0 \\ \sum_{i=0}^{n-1} Df^i(x) & 1 \end{pmatrix}.$$

g will thus be tangentially distal of order 2 if and only if $\inf_n Df^n(x) > 0$ for all x, which happens if and only if f is C^1 conjugate to a rotation, as in 1.5. But g will be tangentially distal *of order* 1 if and only if:

$$\sum_{n=0}^{\infty} Df^n(x) = +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} Df^{-n}(x) = +\infty$$

for all $x \in T^1$.

But this is always true, because if $|\alpha - (p_n/q_n)| < 1/q_n^2$ (e.g. if p_n/q_n is a convergent of α) then an inequality of Denjoy (see e.g. [10], ch. VI) says that:

$$e^{-v} \leqq Df^{\pm q_n} \leqq e^{v},$$

where $v = \mathrm{Var}(\log Df)$.

1.11. Let G be a semisimple Lie group, Γ a cocompact discrete subgroup, and $G = KAN$ an Iwasawa decomposition of G [9], where N denotes the nilpotent part of G. Let $(G/\Gamma, N)$ denote the transformation group, where the action of N on G/Γ is given simply by left multiplication.

When $G = \mathrm{SL}(2, \mathbf{R})$, and

$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbf{R} \right\},$$

$(G/\Gamma, N)$ is a classical horocycle flow, and, in general, $(G/\Gamma, N)$ is called a *generalized horocycle flow*, and is minimal, uniquely ergodic with respect to Haar measure [3], [5], [7], [24].

We claim that $(G/\Gamma, N)$ is tangentially distal of all orders.

Let L_n denote the map:

$$L_n : g\Gamma \mapsto ng\Gamma \qquad (g \in G).$$

Let $\mathfrak{v}$ denote the Lie algebra of C^∞ right-invariant vector fields on G, i.e. $v \in \mathfrak{v}$

if and only if:

$$(DR_g)(v(xg^{-1})) = v(x) \qquad \text{for all } x, g \in G,$$

where $R_g(x) = xg$ $(x, g \in G)$.

Let $\pi : G \to G/\Gamma$ be the natural quotient map.

For $v \in \mathfrak{v}$, define a vector field $\bar{v}$ on G/Γ by $\bar{v}(\pi x) = D\pi(v(x))$ for all $x \in G$. This is well-defined.

$$\bar{\mathfrak{v}} = \{\bar{v} : v \in \mathfrak{v}\} \text{ is invariant under the map: } \bar{v} \mapsto DL_n(\bar{v} \circ L_n^{-1}).$$

Let x_0 denote the identity coset in G/Γ. The compactness of G/Γ implies that there exist constants $0 < m_r \leqq M_r < \infty$ for each r $(1 \leqq r \leqq \dim G)$ such that, for all $v \in \Lambda^r \bar{\mathfrak{v}}$,

$$\begin{aligned} m_r \| v(x_0) \| &\leqq \| v(x) \| \\ &\leqq M_r \| v(x_0) \| \end{aligned}$$

where $\| \quad \|$ denotes a fixed norm on $\Lambda^r(G/\Gamma)$.

Then $(G/\Gamma, N)$ is tangentially distal of order r if and only if, whenever $v \in \Lambda^r \bar{\mathfrak{v}}$ with $v \neq 0$ at one (hence every) point:

$$\inf_n \| \Lambda^r(L_n)((v)(L_n^{-1} x_0)) \| > 0. \tag{1.11.1}$$

But if we identify $T_{x_0}(G/\Gamma)$ with $T_1(G)$ under $D\pi$, hence with the Lie algebra $\mathfrak{g}$ of G, [9], then (1.11.1) becomes:

$$\inf_n \| \Lambda^r(\mathrm{Ad}(n)) w \| > 0, \tag{1.11.2}$$

whenever $w \in \Lambda^r \mathfrak{g}$ with $w \neq 0$, where $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ denotes the Adjoint representation.

But (1.11.2) holds because [9] there is a basis of $\mathfrak{g}$ with respect to which all the matrices of $\mathrm{Ad}(N)$ are lower triangular with 1's on the diagonal.

1.12. *Condition for tangential distality of a horocycle flow associated with an Anosov flow*

Background references for the following are [1], [20].

Let X be a compact 3-dimensional manifold, and let $\{f_t\}$ be a C^2 Anosov flow on X, so that:

$$T_* X = E_u \oplus E_f \oplus E_s,$$

where E_u, E_f, E_s are C^0 subvector bundles of T_*X, invariant under Df_t, E_f is the bundle tanget to $\{f_t\}$, spanned by the infinitesimal generator vector field v_f of $\{f_t\}$, and there exist $a, c > 0$, such that:

$$\|Df_t v\| \geqq ce^{at}\|v\| \qquad \text{for } v \in E_u, \quad t > 0,$$

$$\|Df_t v\| \leqq ce^{-at}\|v\| \qquad \text{for } v \in E_s, \quad t > 0,$$

$$\|Df_t v\| \leqq ce^{-at}\|v\| \qquad \text{for } v \in E_u, \quad t < 0,$$

$$\|Df_t v\| \geqq ce^{at}\|v\| \qquad \text{for } v \in E_s, \quad t < 0.$$

Suppose both E_u and E_s are orientable, spanned by non-vanishing vector fields v_u, v_f. Let $\{g_t\}$ be the flow, differentiable with respect to t ($\{g_t\}$ does exist), with infinitesimal generator v_u. We wish to determine when $(X, \{g_t\})$ is tangentially distal of all orders (by 1.8, this property is not affected by the choice of v_u).

Of course, this question does not make sense unless:

$$(x, t) \mapsto g_t x \text{ is } C^1.$$

So suppose this is true.

For fixed x, t, we have:

(1.12.1) $$g_t x = f_r g_{\mu(t,r,x)} f_{-r} x \qquad \text{for all } r \in \mathbf{R},$$

where $\mu(t, r, x) \to 0$ as $r \to +\infty$.

Write $Df_r v_s(x) = \beta(x, r) v_s(x)$, $(x \in X, r \in \mathbf{R})$ so that $\beta(x, -r) \to \infty$ as $r \to +\infty$.

(1.12.1) is used to obtain (1.12.3), (1.12.4):

(1.12.2) $$Dg_t v_u(x) = v_u(g_t x) \qquad (x \in X, \ t \in \mathbf{R}).$$

(1.12.3) $$Dg_t v_f(x) \wedge v_u(g_t x) = v_f(g_t x) \wedge v_u(g_t x).$$

(1.12.4) For $x \in X$, $t \in \mathbf{R}$, $\lim_{r \to +\infty} \dfrac{\beta(x, -r)}{\beta(g_t x, -r)}$ exists and:

$$Dg_t v_s(x) \wedge v_f(g_t x) \wedge v_u(g_t x) = \left(\lim_{r \to +\infty} \frac{\beta(x, -r)}{\beta(g_t x, -r)}\right) v_s(g_t x) \wedge v_f(g_t x) \wedge v_u(g_t x).$$

Thus $(X, \{g_t\})$ is tangentially distal of all orders if and only if:

$$\inf_t \left(\lim_{r \to +\infty} \frac{\beta(x, -r)}{\beta(g_t x, -r)}\right) > 0 \qquad \text{for all } x \in X.$$

Using 2.2, we have the following (with notation of this section):

1.12.5. Proposition. *Let $(X, \{g_t\})$ be minimal (which happens, for example, if*

$(X,\{f_t\})$ is not the time 1 *suspension of an Anosov diffeomorphism* [20]). *Then $(X,\{g_t\})$ is tangentially distal if and only if both the following hold*:

(i) $(x,t)\mapsto g_t x$ *is* C^1.

(ii) $0<\inf_{\substack{x,y\in X\\ r\in\mathbf{R}}}\frac{\beta(x,r)}{\beta(y,r)}\leqq\sup_{\substack{x,y\in X\\ r\in\mathbf{R}}}\frac{\beta(x,r)}{\beta(y,r)}<\infty.$

Note that (ii) can be formulated without reference to (i). We state the following without proof (using the notation of this section):

PROPOSITION. *Suppose X supports a C^0 f_t-invariant measure μ. The following are equivalent*:

(i) *μ maximizes the entropy of f_1.*

(ii) $0<\inf_{\substack{x,y\in X\\ r\in\mathbf{R}}}\frac{\beta(x,r)}{\beta(y,r)}\leqq\sup_{\substack{x,y\in X\\ r\in\mathbf{R}}}\frac{\beta(x,r)}{\beta(y,r)}<\infty.$

We omit the details, but using this it can be shown that, if X is the unit tangent bundle of a compact surface S with a Riemannian structure, and $((X,\{f_t\})$ is the geodesic flow, and is C^2, then $(X,\{g_t\})$ *cannot* be tangentially distal of all orders unless S has constant curvature.

However we have the following, stated without proof (actually, it is quite easy):

PROPOSITION. *Let $(X,\{f_t\})$ be C^2 Anosov, and suppose E_u, E_s are orientable C^r $(r\geqq 1)$. Then any C^r change of velocity $(X,\{F_t\})$ of $(X,\{f_t\})$ is Anosov, and there exists a C^r change of velocity $(X,\{F_t\})$ such that, if the associated stable and unstable vector bundles are E'_s, E'_u, then these are integrable, and for some vector fields w_s, w_u spanning E'_s, E'_u respectively*:

$$(F_t)_* w_s = e^{-t} w_s,$$

$$(F_t)_* w_u = e^{t} w_u.$$

Thus it would seem that there are probably many Anosov flows satisfying condition (ii) of 1.12.5. The problem is: would these flows also satisfy condition (i)?

§2. A structure theorem for vector bundle distal extensions

We assume (*throughout this section*) that X is a compact metric space, that $f:X\to X$ is a minimal homeomorphism, that V is a real finite-dimensional C^0 vector bundle over X, and that (V,f_v) is an extension of (X,f).

We assume without loss of generality that a fixed "norm" on V arises from a Riemannian structure on V — and the norms used on the bundles $\bigotimes^r V$, $\wedge^r V$ will be those arising from the associated Riemannian structures on these bundles.

We prove a Furstenberg-type structure theorem for those (V, f_v) for which $(\wedge^k V, \wedge^k f)$ is a distal extension of (X, f) for $1 \leqq k \leqq n = \dim V$ (2.8).

The basic result is:

2.1. Proposition. *If $f_v : V \to V$ is a distal extension of $f : X \to X$, then $W = \{w \in V : \sup_{n \in \mathbf{Z}} a \| f_v^n w \| < \infty\}$ is non-trivial.*

Proof. Let V_x denote the fibre of V over x, with zero vector $\mathbf{0}_x$.

Define $T : V \to V$ by:

$$T\mathbf{0}_x = \mathbf{0}_{fx}$$

$$Tw = \frac{f_v w}{\| f_v w \|} \qquad \text{if } w \neq \mathbf{0}_x \text{ for any } x.$$

Then $\| Tw \| = \| w \|$ for all $w \in V$, and $T^n w = f_v^n w / \| f_v^n w \|$ if $w \neq 0$.

Let $w \in V$ have minimal closed orbit C_w under T, $\| w \| = 1$.

$L : C_w \to \mathbf{R}^+$ defined by:

$$L(z) = \inf_{n \in \mathbf{Z}} \| f_v^n z \|$$

is a strictly positive, upper semi-continuous function, hence has a point of continuity w_0. Then there exists an open neighbourhood U of w_0 in C_w such that:

$$L(z) > \frac{L(w_0)}{2} \qquad \text{for all } z \in U.$$

By minimality of (C_w, T), [6, ch. 2], $T^n w_0 \in U$ for all $n \in A$, where A is a syndetic subset of $\mathbf{Z}$ (i.e. $A + F = \mathbf{Z}$ for some finite $F \subseteq \mathbf{Z}$).

So $L(T^n w_0) > L(w_0)/2$ for all $n \in A$, i.e.

$$L(w_0) = L(f_v^n w_0) > \frac{L(w_0)}{2} \| f_v^n w_0 \| \quad \text{for all } n \in A.$$

So $\| f_v^n w_0 \| < 2$ for all $n \in A$. Since A is syndetic, $\sup_n \| f_v^n w_0 \| < \infty$. So W is non-trivial.

2.2. Corollary. *In 2.1, if in addition V is 1-dimensional, there exist* *m*, *M* > 0 *such that*:

$$m\|w\| \leqq \|f_v^n w\| \leqq M\|w\| \quad \textit{for all } w \in V.$$

PROOF. By 2.1, choose w with $\|w\| = 1$ and $\sup_n \|f_v^n w\| = M_1 < \infty$. Put

$$M_2 = \inf_n \|f_v^n w\| > 0.$$

Then

$$\frac{M_2}{M_1}\|f_v^m w\| \leqq \|f_v^{n+m} w\| \leqq \frac{M_1}{M_2}\|f_v^m w\| \qquad \text{for all } n, m \in \mathbf{Z},$$

i.e.

$$\frac{M_2}{M_1} \leqq \|f_v^n(T^m w)\| \leqq \frac{M_1}{M_2}, \qquad \forall n, m \in \mathbf{Z}.$$

Then

$$\frac{M_2}{M_1} \leqq \|f_v^n z\| \leqq \frac{M_1}{M_2} \qquad \text{for all } n \in \mathbf{Z}, \quad z \in C_w.$$

The result follows, since $C_w \cap V_x \neq \emptyset$ for all $x \in X$, and V_x is 1-dimensional.

NOTE. In 1.5 we apply this result to the case when X is the circle, $V = T_* X$, f is C^1 and $f_v = Df$.

2.3. PROPOSITION. *If* $\wedge^k f_v : \wedge^k V \to \wedge^k V$ *is a distal extension of* $f: X \to X$, $1 \leqq k \leqq n = \dim V$, *then for some* $m, M > 0$ *the following sets coincide*:

$$W = \{w \in V : \sup_n \|Df_v^n w\| < \infty\},$$

$$W'_{m,M} = \{w \in V : m\|w\| \leqq \|f_v^n w\| \leqq M\|w\| \quad \textit{for all } n \in \mathbf{Z}\},$$

$$W'' = \{w \in V : (C_w, T) \textit{ is minimal}\}.$$

PROOF. $W = W'_{m,M}$. If $W_x = V_x \cap W$, then W_x is clearly a vector space. Let x_0 be such that:

$$\dim W_{x_0} = \operatorname*{Max}_{x \in X} \dim W_x.$$

We shall find (Lemma 2.4) constants $\alpha, \beta > 0$ such that:

$$\alpha\|w\| \leqq \|f_v^n w\| \leqq \beta\|w\| \qquad \text{for all } w \in W_{x_0}, \qquad n \in \mathbf{Z},$$

which will imply $W = W'_{m,M}$ with $m = \alpha/\beta$ and $M = \beta/\alpha$. For then, given $x \in X$, we can find a sequence $\{n_m\}$ such that $f^{n_m}x_0 \to x$, and $\{f_v^{n_m} w\}$ converges for all $w \in W_{x_0}$, and if $A : W_{x_0} \to V_x$ is defined by:

$$Aw = f_v^{m_n} w,$$

then A is a linear map, one-to-one (because f_v is a distal extension), and since

$$\frac{\alpha}{\beta}\| f_v^{n_m} w \| \leqq \| f_v^{r+n_m} w \| \leqq \frac{\beta}{\alpha}\| f_v^{n_m} w \|$$

for all n_m, $r \in \mathbf{Z}$, for all $w \in W_{x_0}$,

$$\frac{\alpha}{\beta}\| Aw \| \leqq \| f_v^r Aw \| \leqq \frac{\beta}{\alpha}\| Aw \|,$$

so that, *a fortiori* $A(W_{x_0}) \subseteq W_x$, hence $A(W_{x_0}) = W_x$, since dim W_{x_0} is maximal.

$W'' \subseteq W = W'_{m,M}$. If $w \in W''$, the method of proof of 2.1 implies $C_w \cap W \neq \emptyset$. Clearly W is T-invariant and since $W = W'_{m,M}$, W is closed. Hence $C_w \subseteq W$, and $w \in W$.

$W'_{m,M} = W \subseteq W''$. Let $w \in W$, and let w_0 be in the same fibre over X as w, $w_0 \in C_w$, such that (C_{w_0}, T) is minimal. Then $w_0 \in W'' \subseteq W = W'_{m,M}$. It suffices to show $w \in C_{w_0}$.

Let $T^{m_n} w \to w_0$. Then

$$\| T^{-m_n} w_0 - w \| = \| T^{-m_n}(w_0 - T^{m_n} w) \|$$
$$\leqq M \| w_0 - T^{m_n} w \| \to 0. \qquad \text{Q.E.D.}$$

2.4. Lemma. *If* $(\wedge^k V, f_v)$ *is a distal extension of* (X, f) $(1 \leqq k \leqq n)$, *and* $W_x = \{w \in V_x : \sup_n \| f_v^n w \| < \infty\}$ *then there exist* $\alpha, \beta > 0$ *such that*:

$$\alpha \| w \| \leqq \| f_v^n w \| \leqq \beta \| w \| \qquad \textit{for all } w \in W_x.$$

Proof. The existence of β follows immediately from the finite-dimensionality of W_x.

Existence of α. Let dim $W_x = k$, and let $e_1 \cdots e_k$ be an orthonormal basis for W_x. Let $e_1^n \cdots e_k^n$ be an orthonormal basis for $f_v^n(W_x)$. Let A_n denote the matrix of f_v^n with respect to the bases $\{e_1 \cdots e_k\}$ and $\{e_1^n \cdots e_k^n\}$. Clearly $\| A_n \|$ and $\| A_n^{-1} \|$ are independent of the choice of orthonormal bases (where $\| A_n \|$ denotes the operator norm).

$$A_n = (a_{ij}^n) \quad \text{and} \quad A_n^{-1} = (b_{ij}^n), \quad \text{say.}$$

To prove the existence of α, it suffices to prove:

$$\sup_{n\in\mathbf{Z}} \sup_{1\leqq i,j\leqq k} |b_{ij}^n| < \infty,$$

since the left-hand side is $\geqq (1/k)\sup_n \|A_n^{-1}\|$.

$$|a_{ij}^n| = |\langle A_n e_i, e_j^n\rangle| \leqq \beta.$$

So

$$|b_{ij}^n| \leqq \frac{(k-1)!\,\beta^{k-1}}{\det A_n}.$$

So it suffices to show: $\inf_n \det A_n > 0$. But

$$\inf_n \det A_n = \inf_n \|\wedge^k f_v^n(e_1 \wedge \cdot \wedge e_k)\| > 0.$$

2.5. Remark. If $W = \{w \in V : \sup_n \|Df_v^n w\| < \infty\}$, then the proof of 2.3 shows that the dimension of the vector space $W \cap V_x = W_x$ is the same for all x. Also, the fact that $W = W'_{m,M}$ for some m, M, implies that W is closed. It follows that if $\langle\ ,\ \rangle$ is any Riemannian metric, and $P: V \to W$ is the orthogonal projection onto W, then P is continuous. Hence W is an f_v-invariant C^0 subvector bundle.

2.6. Remark. (Obvious but important.) If (V, f_v) is a distal extension of (X, f), and W is an f_v-invariant C^0 subvector bundle of V, then (W, f_v) is a distal extension of (X, f).

2.7. We need the following proposition for the inductive step in the proof of the main structure theorem:

Proposition. *If* $(\wedge^k V, \wedge^k f_v)$ *is a distal extension of* (X, f) $(1 \leqq k \leqq n = \dim V)$, *and* W *is a* C^0 f_v*-invariant subvector bundle of* V, *and* $f_{v/w}: V/W \to V/W$ *is given by*:

$$f_{v/w}(W + u) = W + f_v u,$$

then $(\wedge^k(V/W), \wedge^k f_{v/w})$ *is a distal extension of* (X, f) $(1 \leqq k \leqq n - r,\ r = \dim W)$.

Proof. For all k, let $A_k(V)$ denote the $\bigotimes^k f_v$-invariant subvector bundle of $\bigotimes^k V$ spanned by vectors of the form $v_1 \otimes \cdots \otimes v_k$, where at least two of the v_i's are the same. Recall that $\wedge^k V = \bigotimes^k V / A_k(V)$.

Fix k $(1 \leqq k \leqq n - r)$, $x \in X$, and $u \neq 0$ in the fibre of $\wedge^k(V/W)$ over x. To prove the proposition under the given hypotheses, we need to show:

$$\inf_{m: f^m x \in U} \| \wedge^k (f^m_{v/w}) u \| > 0, \tag{2.7.1}$$

for some open neighbourhood U of x.

Choose U, such that there are continuous cross-sections $v_1 \cdots v_n : U \to V$ with $v_1(y) \cdots v_n(y)$ an orthonormal basis of V_y for all $y \in U$, and $v_1(y) \cdots v_r(y) \in W_y$.

Let $Y = (\bigotimes^r W) \otimes (\bigotimes^k V)/A_{r+k}(V) \subseteq \wedge^{r+k}(V)$, and $f_Y = \wedge^{r+k}(f_V)$ restricted to Y. Let $Z = \wedge^k(V/W)$ and $f_Z = \wedge^k(f_{V/W})$.

Let $\pi_Y : Y \to X$ and $\pi_Z : Z \to X$ be the natural projections.

We define a homeomorphism $\Phi : \pi_Y^{-1}(U) \to \pi_Z^{-1}(U)$, inducing the identity on U, which is an isometric vector space isomorphism restricted to each fibre, by sending the basis element:

$$A_{r+k}(V) + (v_1(y) \otimes \cdots \otimes v_r(y) \otimes v_{i_1}(y) \otimes \cdots \otimes v_{i_k}(y))$$

$$(r < i_1 < \cdots < i_k)$$

in Y_y to the basis element:

$$A_k(V/W) + (W + v_{i_1}(y)) \otimes \cdots \otimes (W + v_{i_k}(y))$$

in Z_y. Define $\lambda_m(x) \in \mathbf{R}$, for m such that $f^m x \in U$, by:

$$\wedge^r(f^m_v)(v_1(x) \wedge \cdots \wedge v_r(x)) = \lambda_m(x)(v_1(f^m x) \wedge \cdots \wedge v_r(f^m x)).$$

By 2.6, $(\wedge^r W, \wedge^r(f_v))$ is a distal extension of (X, f), and since $\wedge^r W$ is 1-dimensional, 2.2 implies that:

$$\sup_{m: f^m x \in U} |\lambda_m(x)| < \infty. \tag{2.7.2}$$

Also, if $v \in Y_x$, and $f^m x \in U$,

$$\Phi(f^m_Y v) = \lambda_m(x) f^m_Z(\Phi v). \tag{2.7.3}$$

By 2.6, (Y, f_Y) is a distal extension of (X, f). So putting $\Phi v = u$ in 2.7.3, and using 2.7.2, we obtain 2.7.1, as required.

2.8. *Structure theorem for extensions which are distal of all orders*

Let X be compact metric, and V a finite-dimensional real vector bundle over X. Let (X, f) be minimal, and $(\wedge^k V, \wedge^k f)$ a distal extension of (X, f) ($1 \leqq k \leqq n = \dim V$). Then there exist f_v-invariant C^0 subvector bundles

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_r = V,$$

such that:

$$V_i = \{w \in V : \sup_m \| f_v^m w + V_{i-1} \| < \infty\}$$

and there exist $m_i, M_i > 0$ such that:

$$m_i \| w + V_{i-1} \| \leqq \| f_v^m w + V_{i-1} \| \leqq M_i \| w + V_{i-1} \| \tag{2.8.1}$$

for all $w \in V_i$.

PROOF. We simply apply 2.3–2.5 successively to the extensions $(V/V_i, f_v)$ of (X, f) for $i = 1, 2, \cdots$. The process terminates after finitely many steps, since V is finite-dimensional.

REMARKS ON 2.8.1. (1) In general, the V_i are not integrable, as the following example shows.

Let $X = SL(3, \mathbf{R})/\Gamma$, for some discrete cocompact subgroup Γ of $SL(3, \mathbf{R})$. Let N be the subgroup

$$\left\{ \begin{pmatrix} 1 & s & t \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} : s, t, u \in \mathbf{R} \right\} \quad \text{of } SL(3, \mathbf{R}).$$

$(N, SL(3, \mathbf{R})/\Gamma)$ is minimal tangentially distal (1.11). Using the notation of 1.11, $\mathfrak{v}$ is isomorphic, as a Lie algebra, to $SL(3, \mathbf{R})$ the Lie algebra of 3×3 real matrices with zero trace. Each V_i is C^∞, and is spanned by elements of $\mathfrak{v}$, and the subspace of $SL(3, \mathbf{R})$ corresponding to V_4 is $\{(x_{ij}) : x_{31} = 0\}$, which is not a Lie algebra. Hence, by Frobenius' theorem [12], V_4 is not integrable.

(2) The converse to the structure theorem is true. Suppose (X, f) is minimal, and (V, f_v) is an extension of (X, f), where V is a finite-dimensional real vector bundle over X, and there exist f_v-invariant C^0 subvector bundles $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V$, and constants $m_i, M_i > 0$ such that

$$m_i \| w + V_{i-1} \| \leqq \| f_v^r (w + V_{i-1}) \| \leqq M_i \| w + V_{i-1} \|$$

for all $w \in V_i$, all r.

Then $(\Lambda^k(V), \Lambda^k(f_v))$ is a distal extension of (X, f) $(1 \leqq k \leqq n = \dim V)$.

§3. Properties of tangentially distal diffeomorphisms

In this section it is shown that tangential distality of orders 1 and n (where n

denotes the dimension of the manifold) implies zero entropy for a diffeomorphism of a C^∞ manifold. (The diffeomorphism need not be C^∞.) I am indebted to M. Herman for pointing out that this is not dependent on the structure theorem of §2.

It is also shown, in the second half of this section, using the general theory of §2, how tangential distality, *with* certain additional assumptions, implies distality.

3.1. Let X be a compact C^1 manifold, and $f : X \to X$ a C^1 diffeomorphism. Let $h(f)$ denote the topological entropy of f, and, for an f-invariant probability measure μ on X, let $h(f,\mu)$ denote the measure-theoretic entropy with respect to μ.

As is well-known,

$$h(f) = \sup_{\substack{\mu \text{ ergodic} \\ f\text{-invariant}}} h(f,\mu).$$

3.2. We state the following theorem, quoted in [23], which is attributed, with modifications, to Oseledec [16] and Raghunathan [21].

THEOREM. *Let X be a compact C^1 manifold, $f : X \to X$ a C^1 diffeomorphism, and μ an f-invariant probability measure on X. Fix a Riemannian structure on X. Then there exists $\Omega \subseteq X$ with $\mu(\Omega) = 1$, and a measurable section*

$$x \mapsto A_x \qquad (x \in \Omega)$$

of the bundle with fibre $L(T_xX)$ over x, where $L(T_xX)$ denotes the space of linear operators of T_xX into itself, such that A_x is a positive linear operator on T_xX, and:

$$\lim_{m\to\infty} (Df^m_{f^m x} Df^m_x)^{1/2m} = A_x \qquad \textit{for all } x \in \Omega.$$

Moreover, if e^{λ_x} is an eigenvalue of A_x (possibly with $\lambda_x = -\infty$) then there exists $v \in T_xX$ such that:

$$\lim_m \frac{1}{m} \log \| Df^m_x v \| = \lambda_x.$$

It follows from [23], theorem 2, that if X is a C^∞ manifold, and if, for almost all $x \in \Omega$, all the eigenvalues of A_x are 1_x (i.e. A_x = identity almost everywhere) then $h(f,\mu) = 0$.

Thus, in order to show (X,f) has zero entropy if (X,f) is tangentially distal of orders 1 and n, we shall show that, if $\{A_x\}$ is the family of positive operators associated with an ergodic non-atomic f-invariant probability measure μ, then for almost all x, all the eigenvalues of A_x are 1.

3.3. LEMMA. *If (X, f) is tangentially distal of order 1, and μ is an f-invariant probability measure on X, then with the notation of 3.2, all eigenvalues of A_x are $\geqq 1$, for all $x \in \Omega$.*

PROOF. If $\lambda_x < 0$ (i.e. $e^{\lambda_x} < 1$), then:

$$\lim_{m \to \infty} \frac{1}{m} \log \| Df_x^m v \| = \lambda_x < 0, \qquad \text{for some } v \in T_x X,$$

i.e.

$$\lim_{m \to \infty} \log \| Df_x^m v \| = -\infty,$$

i.e.

$$\lim_{m \to \infty} \| Df_x^m v \| = 0,$$

contradicting tangential distality.

3.4. LEMMA. *Let μ be an f-invariant non-atomic ergodic probability measure on X. Let $\dim X = n$, and let $\wedge^n(A_x)$, $\wedge^n(Df_{f^m x}^{m*} Df^m x)$ denote the nth exterior products of A_x, $(Df_{f^m x}^m)^* Df_x^m$ respectively, which are the mappings of $\wedge^n(T_* X)$.*

Then $\wedge^n(A_x) =$ identity on $\wedge^n(T_x X)$ for almost every $x \in \Omega$, if (X, f) is tangentially distal of order n.

PROOF. Clearly, from 3.2,

$$\begin{aligned} \| \wedge^n(A_x) \| &= \lim_{m \to \infty} \| \wedge^n((Df^m)^*(Df_x^m))^{1/2m} \| \quad \text{for all } x \in \Omega \\ &= \lim_{m \to \infty} \| \wedge^n(Df_{f^m x}^m)^* . \wedge^n(Df_x^m)\}^{1/2m} \| \\ &= \lim_{m \to \infty} \| \wedge^n(Df_x^m) v \|^{1/m} \quad \text{for all } v \in \wedge^n(T_x X) \text{ with } \| v \| = 1 \\ &\geqq 1 \quad \text{by tangential distality of order } n. \end{aligned}$$

So $\| \wedge^n(A_x) \| = 1$ if there exists a sequence $\{m_r\}$ such that $m_r \to \infty$ and:

$$\sup_r \| \wedge^n(Df_x^{m_r}) v \| < \infty \quad \text{for } v \in \wedge^n(T_x X) \text{ with } \| v \| = 1.$$

Thus, to complete the proof of 3.4, it suffices to prove:

3.5. LEMMA. *If X is a compact C^1 manifold and f a C^1 diffeomorphism, and μ is an f-invariant non-atomic probability measure on X, and if (V, f_v) is a* distal *C^0 vector bundle extension of (X, f), with V* 1-dimensional *in each fibre, then there exists $\Omega_1 \subseteq X$ with $\mu(\Omega_1) = 1$ such that if $x \in \Omega_1$, there exists a strictly increasing sequence $\{m_r\}$ such that*:

$$\sup_r \| f_v^{m_r} w \| < \infty \qquad \textit{for all } w \in V_x.$$

PROOF. Let $L(x) = \inf_m \| f_v^m w \|$ for $w \in V_x$, $\| w \| = 1$. (This is well-defined, since V is 1-dimensional.)

Given $\varepsilon > 0$, there exists a compact $K \subseteq X$ such that $L | K$ is continuous and $\mu(K) > 1 - \varepsilon$.

So there exist $m, M > 0$ such that:

$$m \leqq L(x) \leqq M \qquad \text{whenever } x \in K.$$

Let $w \in V_x$, $\| w \| = 1$.

$$\| f_v^n w \| L(f^n x) = L(x).$$

So if $x, f^n x \in K$,

$$m \| f_v^n w \| \leqq L(x) \leqq M.$$

So $\| f_v^n w \| \leqq M/m$.

So a sequence $\{m_r\}$, as in the statement of the lemma, exists for almost all $x \in K$ (a different sequence, possibly, for different x), provided that

$$\mu\{x \in K : f^n x \in K \text{ for infinitely many } n\} = \mu(K).$$

But this follows from the Poincaré recurrence theorem.

Hence a sequence $\{m_r\}$ exists for almost all $x \in K$.

Since $\mu(K) > 1 - \varepsilon$ and ε is arbitrary, the lemma is proved.

3.6. THEOREM. *If (X, f) is tangentially distal of orders* $1, n$ $(= \dim X)$ *f has zero topological entropy.*

PROOF. Let μ be an f-invariant ergodic non-atomic probability measure on X, and $\{A_x\}$ the positive operators associated with μ as in 3.2. As stated previously, it suffices to show that for almost every x, the eigenvalues of A_x are all 1. By 3.4, $\det A_x = 1$ a.e. x.

So eigenvalues of $A_x < 1$ occur if and only if eigenvalues > 1 occur. By 3.3, this cannot happen. Q.E.D.

3.7. As before, X is a compact C^1 manifold and f is a minimal C^1 diffeomorphism, tangentially distal of all orders, and:

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = T_* X,$$

where the V_i are the Df-invariant C^0 subvector bundles with:

$$V_i = \{v \in T_* X : \sup_n \| Df^n v + V_{i-1} \| < \infty\}.$$

Let $\pi : T_* X \to X$ be the canonical quotient map. Recall that V_i is *integrable* at $x \in X$ if there exists a C^1 chart (U, φ) at x (i.e. U is open in $\mathbf{R}^n$ and φ maps U diffeomorphically onto a neighbourhood of x) such that:

$$D\varphi : \pi^{-1}(\varphi^{-1} U) \to U \times \mathbf{R}^n$$

maps $V_i \cap \pi^{-1}(\varphi^{-1} U)$ to $U \times \mathbf{R}^k \times \{0\}$, if V_i is k-dimensional in each fibre.

If V_i is integrable, recall that the *leaves* of V_i are the connected components of X in the topology $\mathfrak{T}_1$, where, if x, U, φ are as above, then a $\mathfrak{T}_1$-neighbourhood base at x is given by:

$$\{\varphi^{-1}(W \times \{y_2\}) : W \text{ is a neighbourhood of } y_1 \text{ in } \mathbf{R}^k\}$$

where $\varphi(x) = (y_1, y_2)$.

In what follows, we find a condition for the distality of (X, f).

3.8. Proposition. (i) *If V_1 is integrable, and one leaf of V_1 is $\mathfrak{T}_1$-compact (and hence compact in the usual topology of X) then all leaves of V_1 are compact, and there exists a C^1 manifold X_1 and a C^1 submersion $\rho_1 : X \to X_1$ with* $\operatorname{Ker} D\rho_1 = V_1$, *and $\{\rho_1^{-1}(y) : y \in X_1\}$ are the leaves of V_1.*

(ii) *Let $f_1 : X_1 \to X_1$ be the diffeomorphism which is well-defined by:*

$$\rho_1(fx) = f_1(\rho_1 x).$$

Then (X, f) is an almost periodic extension of (X_1, f_1), i.e. let d be a metric on X. Then given $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\rho_1(x) = \rho_1(y)$ and $d(x, y) < \delta$, then $d(f^n x, f^n y) < \varepsilon$ for all n.

Proof. Palais [17] has shown that X_1, ρ_1 exist with the required properties (i) provided that V_1 is *regular*. V_1 is *regular at $x \in X$* if the chart (U, φ) of 3.7 can be chosen such that if:

$$U_y = \{w \in \mathbf{R}^k : (w, y) \in U\},$$

then $\varphi^{-1}U_y$ and $\varphi^{-1}U_z$ belong to different leaves of V_1 whenever U_y, U_z are non-empty with $y \neq z$. V_1 is ***regular*** if it is regular at all points.

If $w \in X$, let L_w denote the leaf of V_1 containing w. Let $x \in X$ be such that L_x is compact. Since f is minimal, to prove (i) of the proposition, it suffices to prove V_1 is regular at x.

Let $\{((-1,1)^n, \varphi_j) : j = 1, \cdots, m\}$ be a set of integrable charts of V_1, as in 3.7, with

$$X \subseteq \varphi_1^{-1}(-1/2, 1/2)^n \cup \cdots \cup \varphi_m^{-1}(-1/2, 1/2)^n,$$

and $x \in \varphi_1^{-1}(-1/2, 1/2)^n$.

Proof of (i). *Step 1.* $L_x \subseteq \bigcup_{j=1}^{s} \varphi_{i_j}^{-1}((-1/2, 1/2)^k \times \{y_{i_j}\})$. We show there exists $N > 0$ such that each leaf $f^l(L)$ intersects the sets $\varphi_i^{-1}((-1,1)^n)$ in at most N slices altogether.

Let $\|D_1(\varphi_j \circ f^l \circ \varphi_i^{-1})\| \leqq M$ for all i, j, l, where D_1 denotes the derivative with respect to the first k coordinates, and M is an integer.

Divide each set $(-1/2, 1/2)^k \times \{y_{i_j}\}$ into $(2M)^k$ cubes of side $1/2M$.

For each integer l, the image under f^l of one of the cubes of side $1/2M$ is contained completely in:

$$\varphi_j^{-1}((-1,1)^k \times \{z\})$$

for some j, and some $z \in (-1/2, 1/2)^{n-k}$.

Hence $f^l(L_x)$ is covered by a set of at most $s \times (2M)^k$ sets of the form $\varphi_j^{-1}((-1,1)^k \times \{z\})$. Any such set has non-null intersection with at most $m-1$ other sets $\varphi_i^{-1}((-1,1)^k \times \{z'\})$.

So $f^l(L_x)$ intersects at most $ms \times (2M)^k = N$ sets of the form $\varphi_j^{-1}((-1,1)^k \times \{z\})$.

Proof of (i). *Step 2.* *Any* leaf L intersects at most N sets of the form $\varphi_j^{-1}((-1,1)^k \times \{z\})$ (so that any leaf is compact). This follows from the fact that this is true for the dense set of leaves $\{f^l(L_x)\}$.

Proof of (i). *Step 3.* Step 2 implies the following:

(3.8.1) Let d be a metric on X. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(w, z) < \delta$, then $d(w', L_z) < \varepsilon$ for all $w' \in L_w$.

Now choose a neighbourhood $U_1 \subseteq \varphi_1^{-1}((-1,1)^n)$ of x such that $\varphi_1(L_x)$ intersects $\varphi_1(U_1)$ in only one slice. (3.8.1) implies we can choose U_1 such that *all* leaves intersect U_1 in at most one slice.

Hence V_1 is regular at x, as required.

Proof of (ii). By minimality of f, it suffices to prove that given $\varepsilon > 0$ there exists $\delta > 0$ such that, whenever $y_1, y_2 \in \varphi_1^{-1}((-1/2M, 1/2M)^n)$ with $p_2(\varphi_1 y_1) = p_2(\varphi_1 y_2)$ (where p_2 denotes projection onto the last $n-k$ coordinates), and $d(y_1, y_2) < \delta$, then $d(f^l y_1, f^l y_2) < \varepsilon$ for all l.

As in step 1 of the proof of (i), if

$$y_1, y_2 \in \varphi_1^{-1}((-1/2M, 1/2M)^n) \quad \text{with } p_2\varphi_1 y_1 = p_2\varphi_1 y_2,$$

then for each l, $f^l\varphi_1^{-1}(-1/2M, 1/2M)^n$ lies completely in $\varphi_j^{-1}((-1,1)^k \times \{w\})$ for some j, w, and

$$\| D_1(\varphi_j \circ f^l \circ \varphi_1^{-1}) \| \leqq M \quad \text{on } (-1/2M, 1/2M)^n.$$

Hence, for all l,

$$\| \varphi_j \circ f^l \circ \varphi_1^{-1}(y_1) - \varphi_j \circ f^l \circ \varphi_1^{-1}(y_2) \| \leqq M \| y_1 - y_2 \|.$$

The result follows.

3.9. Corollary. *If X, f, $V_1, \cdots, V_r$ are as in 3.7, and each V_i is integrable with compact leaves then there exist C^1 manifolds X_i $(i = 1, \cdots, r)$ and C^1 diffeomorphisms $f_i : X_i \to X_i$ and C^1 submersions $\rho_i : X_i \to X_{i+1}$ with $\rho_i \circ f_i = f_{i+1} \circ \rho_i$ and $V_i = \operatorname{Ker} D(\rho_i \circ \cdots \circ \rho_1)$. Moreover (X_i, f_i) is an almost periodic extension of (X_{i+1}, f_{i+1}) and hence (since (X_r, f_r) is trivial), (X, f) is distal.*

Proof. Use induction and 3.8. Note that if V_i is integrable with compact leaves for X, then $D(\rho_{i-1} \circ \cdots \circ \rho_1) V_i$ is integrable, with compact leaves, for X_{i-1}.

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